Modal Analysis Summarized

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Rev. 1.2
**Modal Analysis**

It is the process of determining the inherent system dynamical characteristics such as modal frequencies, damping factors and mode shapes, and using these parameters in constructing a mathematical model. The mathematical model is called Modal Model and the dynamical characteristics are referred as modal data.

**Modal Testing**

The experimental technique used to drive modal model assuming linear time-invariant dynamic system is referred as modal testing. The relation between the excitation at a certain point and the response at the same point or other points, namely the Frequency Response Functions, or FRFs are targeted in model testing. Combinations of relations between excitations and responses at different locations constitute a complete set FRFs, normally arranged in a matrix.

**Single Degree of Freedom System SDoF**

The theory for an SDoF system is the basis for the analysis of a system with more than one DoF. It also provides physical insight into the vibration of a structural system.

![Figure (1) Single Degree of Freedom System](image)

Equations of motion are given by

\[
m \dddot{x} + c \dot{x} + kx = f(t) \quad \text{for viscous damping} \quad (1)
\]

\[
m \dddot{x} + jhx + kx = f(t) \quad \text{for structural damping} \quad (2)
\]

Using Laplace transform:
For viscous damping: \[ X(s) = \frac{F(s)}{ms^2 + cs + k} + \frac{(ms + c)x(0) + m\dot{x}(0)}{ms^2 + cs + k} \] (3)

For structural damping: \[ X(s) = \frac{F(s)}{ms^2 + jh + k} + \frac{mx(0)s + m\dot{x}(0)}{ms^2 + jh + k} \] (4)

Assuming harmonic functions for both \( x(t) \) and \( f(t) \) with frequency \( \omega \) and zero initial conditions, the Frequency Response Function (FRF) of a SDoF can be written as \( (s \rightarrow j\omega) \):

For viscous damping: \[ H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{k - \omega^2 m + j\omega c} \] (5a)

or: \[ H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1/m}{\omega_0^2 - \omega^2 + 2j\zeta\omega_0\omega} \] (5b)

Or more compactly: \[ H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1/k}{1 - \frac{\omega^2}{\omega_0^2} + 2j\zeta\omega_0\omega} \] (5c)

For structural damping: \[ H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{k - \omega^2 m + jh} \] (6a)

or: \[ H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1/m}{\omega_0^2 - \omega^2 + j\eta\omega_0^2} \] (6b)

Or more compactly: \[ H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1/k}{1 - \frac{\omega^2}{\omega_0^2} + j\eta} \] (6c)
The FRF $H(\omega)$ is sometimes denoted as $\alpha(\omega)$. This FRF uses displacement as the response. It is known as Receptance FRF. By replacing the displacement response $\dot{x}(\omega)$ with velocity $X(\omega)$ and acceleration $\ddot{x}(\omega)$, two different types of FRFs can be defined as:

**Mobility FRF:**

$$Y(\omega) = \frac{\dot{X}(\omega)}{F(\omega)} = \frac{j\omega}{k - \omega^2m + j\omega c}$$

(7)

**Accelerance FRF:**

$$A(\omega) = \frac{\ddot{X}(\omega)}{F(\omega)} = \frac{-\omega^2}{k - \omega^2m + j\omega c}$$

(8)

### Circularity of FRF Nyquist Plots

Nyquist plot of FRF is obtained by representing the real part on x-axis and imaginary part on y-axis with frequency as implicit variable. A useful property of FRF Nyquist plot is its circularity. If we consider Mobility FRF in viscous damping, $Y(\omega)$:

$$\text{Re}(Y(\omega)) = \frac{\omega^2c}{(k - \omega^2m)^2 + (\omega c)^2}$$

(9)

$$\text{Im}(Y(\omega)) = \frac{\omega(k - \omega^2m)}{(k - \omega^2m)^2 + (\omega c)^2}$$

(10)

It is easy to show that:

$$\left(\text{Re}(Y(\omega)) - \frac{1}{2c}\right)^2 + \left(\text{Im}(Y(\omega))\right)^2 = \left(\frac{1}{2c}\right)^2$$

(11)

The above equation is equation of a circle with radius $\frac{1}{2c}$ and center $(\frac{1}{2c}, 0)$. Neither receptance nor accelerance will show the same exact circularity. There will be skewness from perfect circularity.
For structural damping, the receptance FRF will be perfect circle while other FRFs will have some skewness:

\[ \text{Re}(\alpha(\omega)) = \frac{k - \omega^2 m}{(k - \omega^2 m)^2 + h^2} \]  

(12a)

\[ \text{Im}(\alpha(\omega)) = \frac{-h}{(k - \omega^2 m)^2 + h^2} \]  

(12b)

It can be easily shown that:

\[ \left( \text{Re}(\alpha(\omega)) \right)^2 + \left( \text{Im}(\alpha(\omega)) + \frac{1}{2h} \right)^2 = \left( \frac{1}{2h} \right)^2 \]  

(13)

The above equation is equation of a circle with radius \(\frac{1}{2h}\) and center \((0, -\frac{1}{2h})\). Neither mobility nor accelerance will show the same exact circularity.

**Multi Degree of Freedom (MDoF) System**

The equation of motion of damped forced multi degree of freedom system is given by:
\( \mathbf{M} \{ \ddot{x} \} + \mathbf{c} \{ \dot{x} \} + \mathbf{k} \{ x \} = \{ f(t) \} \)  

(14)

The Laplace transform of the above equation is:

\[
\left[ \mathbf{M} s^2 + \mathbf{c} s + \mathbf{k} \right] \{ X(s) \} = \{ F(s) \}
\]

(15)

The dynamic stiffness matrix is \( \mathbf{Z}(s) = \left[ \mathbf{M} s^2 + \mathbf{c} s + \mathbf{k} \right] \)

Receptance matrix is

\[
\{ X(s) \} = \mathbf{a}(s) \{ F(s) \} = \left[ \begin{array}{ccc}
\alpha_{11}(s) & \alpha_{12}(s) & \cdots \\
\alpha_{21}(s) & \alpha_{22}(s) & \\
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(16)

Where \( \alpha_{ij}(s) \) relates the response at node \( i \) with the only force at node \( j \).

\[
\mathbf{a}(s) = \left[ \mathbf{M} s^2 + \mathbf{c} s + \mathbf{k} \right]^{-1} = \frac{\text{adj} \left( \mathbf{M} s^2 + \mathbf{c} s + \mathbf{k} \right)}{\det(\mathbf{M} s^2 + \mathbf{c} s + \mathbf{k})}
\]

(18)

When the damping is small, the roots of the characteristic polynomial equation in the numerator are complex conjugate pole pairs, \( \lambda_r \) and \( \lambda_r^* \), \( r = 1, 2, \ldots, N \), with \( N \) the number of modes of the system. The transfer function can be rewritten in a pole-residue form, or the so-called “modal” model (assuming all poles have multiplicity one):

\[
\mathbf{a}(s) = \sum_{r=1}^{N} \left( \frac{\mathbf{R}_r}{s - \lambda_r} + \frac{\mathbf{R}_r^*}{s - \lambda_r^*} \right)
\]

(19)

The residue matrices \( \mathbf{R}_r \) are given by:

\[
\mathbf{R}_r = \lim_{s \to \lambda_r} \left( \mathbf{a}(s)(s - \lambda_r) \right)
\]

(20)

With

\[
\alpha_{ij}(s) = \sum_{r=1}^{N} \left( \frac{r \mathbf{R}_{ij}}{s - \lambda_r} + \frac{r \mathbf{R}_{ij}^*}{s - \lambda_r^*} \right)
\]

(21)

The elements of FRFs are found by letting \( s = j \omega \):
\[ a_{ij}(\omega) = \sum_{r=1}^{N} \left( \frac{r R_{ij}}{j \omega - \bar{\lambda}_r} + \frac{r R_{ij}^*}{j \omega - \bar{\lambda}_r^*} \right) \]  

(22)

The equivalent non-factorized (polynomial) form is given by

\[ \alpha_{ij}(\omega) = \sum_{r=1}^{N} \frac{r A_{ij}}{\omega_r^2 - \omega^2 + 2 j \zeta_r \omega \omega_r} = \sum_{r=1}^{N} \frac{\phi_r \phi_r^*}{\omega_r^2 - \omega^2 + 2 j \zeta_r \omega \omega_r} \]  

(23)

\[ A_r = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix}_m \]  

(24)

Where \( \phi_1, \phi_2, ... \) are modal constants. Shortly we will see these are scaled (or mass–normalized) modal constants. In experimental modal testing, modal constants \( r A_{ij} \) are estimated from the measured FRF data.

Here \( \lambda_m \) is the complex frequency (Eigen value) of the \( m \)th mode:

\[ \lambda_r = -\zeta_r \omega_r + j \sqrt{1 - \zeta_r^2 \omega_r^2} = -\sigma_r + j \omega_{d,r} \]  

(25)

Since the structural damping model is similar to undamped model but with complex stiffness \( K_c = k + j h \), the FRFs can be written as (for structural damping only):

\[ a_{ij}(\omega) = \sum_{r=1}^{N} \left( \frac{r A_{ij}}{\lambda_r^2 - \omega^2} \right) \]  

(26)

However, we should bear in mind that both the modal constants and \( \lambda_m \) are complex quantities.

\[ \lambda_r^2 = \frac{k_r + j h_r}{m_r} = \frac{k_r}{m_r} \left( 1 + j \eta_r \right) = \omega_r^2 \left( 1 + j \eta_r \right) \]  

(27)

Here \( \eta_r \) is defined as structural damping loss factor of the \( r \)th mode.

It worth mentioning that perhaps the most significant outcome of having non-proportional damping is complex vibration modes. The concept of mode shapes was well interpreted in the case of no damping with clearly defined physical meanings. When mode shapes become complex, points on a structure no longer move in a clear pattern of either in or out of phase. Mathematically, we are able to interpret the origins of complex modes. If a dynamic
structure is modeled as an MDoF system with mass, stiffness and damping matrices, then it is
easy to see that the eigenvectors of the system, which are the mode shapes, only become
complex if:
(a) One or more of the matrices are not symmetric, or
(b) The damping matrix is not diagonalizable using the undamped mode shapes.

**SDoF Revisited**  
We must differentiate between *modal constants* and *residues*. Using the SDoF FRF given in eq. (5b):

\[
H(\omega) = \frac{1/m}{\omega_0^2 - \omega^2 + 2\jmath\zeta\omega_0} = \frac{A}{\omega_0^2 - \omega^2 + 2\jmath\zeta\omega_0}
\]

The modal constant for SDoF is equivalent to \(1/m\). While when using partial fractions to factorize the denominator, the FRF can be written as:

\[
H(\omega) = \left( \frac{R}{j\omega - \lambda} + \frac{R^*}{j\omega - \lambda^*} \right) \tag{28}
\]

With residue given by the following equation:

\[
R = \frac{1}{2jm\omega_0} \tag{29}
\]

This can be generalized to MDoF systems. The calculated constants from the measured FRF data depend on the assumed model whether it is partial fractions or non-factorized polynomial.

For MDoF system, the FRF elements are given by

\[
a_{ij}(\omega) = \sum_{r=1}^{N} \left( \frac{r R_{ij}}{j\omega - \lambda_r} + \frac{r^* R_{ij}^*}{j\omega - \lambda_r^*} \right) = \sum_{r=1}^{N} \left( \frac{q_r \phi_{ir} \phi_{jr}}{j\omega - \lambda_r} + \frac{q_r^* \phi_{ir}^* \phi_{jr}^*}{j\omega - \lambda_r^*} \right) \tag{30}
\]

Where \(q_r\) are scaling factors and can be calculated from the following equation for unit modal mass (UMM):

\[
q_r = \frac{1}{2j\omega_r} \tag{31}
\]

**Modal Mass and Modal Stiffness**

Defining mode shape matrix

\[
\psi = \left[ \{\psi\}_1, \{\psi\}_2, \ldots, \{\psi\}_n \right] \tag{32}
\]
Where \( \psi \) represents the mode shape vector for mode \( i \). Note that the mode shape is unique in shape but its values are not because they can be arbitrarily scaled to any set of values.

The modal mass and modal stiffness can be found by post multiplying mass matrix and stiffness matrix by the mode shape matrix and pre multiply them by the transpose of mode shape matrix. They are diagonal matrices.

\[
\psi^T M \psi = \begin{bmatrix}
m_1 & 0 & \cdots & 0 \\
0 & m_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m_n
\end{bmatrix} = \begin{bmatrix}
m_r \\
\end{bmatrix}
\tag{33}
\]

\[
\psi^T K \psi = \begin{bmatrix}
k_1 & 0 & \cdots & 0 \\
0 & k_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & k_n
\end{bmatrix} = \begin{bmatrix}
k_r \\
\end{bmatrix}
\tag{34}
\]

Both the modal mass and modal stiffness matrices are complex and they do not have the same units of the normal mass or normal stiffness. The roots of the system can be found by multiplying the modal stiffness by the inverse of modal mass matrix.

\[
\lambda_r = \left[k_r \right]^{-1}
\tag{35}
\]

They are also sometimes named generalized mass and generalized stiffness matrices.

The mass-normalized mode shape \( \phi \) can be found as follows:

\[
\phi = \frac{1}{\sqrt{m_r}} \psi
\tag{36}
\]

Note that mass-normalized mode shapes are unique in shapes and values because the modal mass values are actually scaling factors since these values are determined by the chosen values of \( \psi \).

Using the mass normalized mode shape matrix, we can prove that:

\[
\Phi^T M \Phi = I
\]

\[
\Phi^T K \Phi = \lambda_r
\tag{37}
\]

From which one can prove eqs. (23) and (26).

Also, modal damping matrix is given by
\[
\Phi^T C \Phi = \begin{bmatrix} c_r \end{bmatrix}
\]  

(38)

Effective and Generalized Mass and Stiffness

As mentioned early, modal mass matrix is not unique as it depends on the selected values of mode shape vectors. Using the mass-normalized mode shape vectors, we can see how to derive quantities which provide us with information about the effective mass (or stiffness) at any point on the structure, such as DOF \( j \). Since the mass-normalized mode shape vectors are unique, the effective mass and stiffness properties are unique too.

The element of effective mass at DOF \( j \) for mode \( r \) can be evaluated as follows:

\[
(m_{e_{jj}})_r = \frac{1}{(\phi_j)_r^2}
\]

(39)

And effective stiffness at node \( j \) for mode \( r \) is given by:

\[
(k_{e_{jj}})_r = \frac{\omega_r^2}{(\phi_j)_r^2}
\]

(40)

Although there is no universal agreement of the definitions of these properties, that which is adopted in this work is to define the generalized mass (or stiffness) of the \( r \)th mode as the effective mass (or stiffness) at the DOF with the largest amplitude of response.

Nyquist plot for MDoF System

The circularity property does not exactly apply for MDoF since any vibration mode will be influenced by a number of modes. However, at the vicinity of a prominent vibration mode, we can assume that the FRF is dominated only by that mode. Thus, the Nyquist plot of SDoF theory can be used in the analysis. This opens an avenue for expanding the SDoF modal analysis method into the MDoF domain.

Frequency Response Function Measurement

Is the quantitative measure of system response to an excitation. FRF is a characteristic of a system that has a measured response resulting from a certain applied input. To measure the frequency response of a mechanical system, both spectra of the input force to the system and the vibration response must be measured. This can be done with a multi-channel FFT Analyzer.
FRF is assumed constant like system parameters (mass, stiffness and damping) and can be formulated using $H_1$, $H_2$ or $H_v$.

There are many testing configurations used in FRF testing:

(a) Single Input - Single Output (SISO)

- The only available option if we have two channel analyzer
- Time invariance problem may exist
- Longest testing time due to roving input and roving output.

(b) Single Input - Multiple Output

- Multi-channel analyzer is required
- Consistent frequency and damping acquired simultaneously
- Shorter testing time than SISO. Roving input still needed.
- Time invariance problem may exist between measurements due to different inputs

(c) Multiple Input - Single Output

- Multi-channel analyzer is required
- Roving measuring transducer is needed.
- Time invariance problem may exist between different measurements

(d) Multiple Input - Multiple Output

- Larger number of channels is required
- Least testing time
- Consistent frequency and damping acquired simultaneously
- Best configuration

**FRF Estimation**

FRF between each input and each output have to be detected to form a matrix of FRFs. FRF calculations are performed in frequency domain to facilitate computations. FFTs of the inputs and outputs are used. Leakage and noise problems may impact the accuracy of calculations.

For input $F_q$ and output $X_p$, the relation is given by:

$$X_p = H_{pq} F_q$$  \hspace{1cm} (41)

Generally, for a system with $M$ inputs and $N$ outputs:
Example of a system with two inputs and two outputs:

\[
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_N \\
\end{bmatrix}
= 
\begin{bmatrix}
H_{11} & H_{12} & \cdots & H_{1M} \\
H_{21} & \cdots & \cdots \\
\vdots & \vdots & \vdots \\
H_{N1} & \cdots & H_{NM} \\
\end{bmatrix}
\begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_M \\
\end{bmatrix}
\]

(42)

Figure (3) Plate with two response and two force locations

Another example is the two plane rigid rotor balancing problem.

Three algorithms exist to estimate FRF referred as \( H_1 \), \( H_2 \) and \( H_v \). The following table lists the main differences between these algorithms:

<table>
<thead>
<tr>
<th>Technique</th>
<th>Solution method</th>
<th>Assumed location of noise</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Input</td>
</tr>
<tr>
<td>( H_1 )</td>
<td>Least squares</td>
<td>No noise</td>
</tr>
<tr>
<td>( H_2 )</td>
<td>Least squares</td>
<td>noise</td>
</tr>
<tr>
<td>( H_v )</td>
<td>Total least squares</td>
<td>noise</td>
</tr>
</tbody>
</table>

\( H_1 \) method assumes no noise at input (force) and tries to reduce noise at the outputs (measured quantities) while \( H_2 \) assumes no noise at output and minimizes noise at the inputs (forces). \( H_v \) method minimizes noise at both inputs and outputs.

**Single Input FRF estimation**
Where:

\[ F = \hat{F} - \nu \]: is the actual force (input)

\[ X = \hat{X} - \mu \]: is the actual response (output)

\( \hat{F} \): is the measured force (measured input) spectrum

\( \hat{X} \): is the measured response (measured output) spectrum

\( \nu, \mu \): the input and output noise spectra respectively

if \( \nu = \mu = 0 \), then the theoretical (expected) FRF is estimated. Otherwise, the Least Squares method is applied to minimize noise and estimate the best FRF in the presence of noise. In order to develop an estimation of FRF, a number of spectra are averaged together in the cross and auto spectrum calculations:

Cross power spectrum of output \( X_p \) and input \( F_q \) is given by:

\[
G_{XF_{pq}} = \sum_{i=1}^{N_{eq}} X_p F_q^* \tag{43}
\]

Auto power spectrum of output \( X_p \) is given by:

\[
G_{XX_{pp}} = \sum_{i=1}^{N_{eq}} X_p^* X_p^* \tag{44}
\]
In fact, the auto power spectrum of a certain signal is the FFT amplitude SQUARED spectrum.

H₁ algorithm minimizes noise at outputs:

\[ H_{pq} = \frac{GXF_{pq}}{GFF_{qp}} \quad (45a) \]

H₂ algorithm minimizes noise at inputs:

\[ H_{pq} = \frac{GXX_{pp}}{GFX_{qp}} \quad (45b) \]

Coherence function can be used to assist the quality of FRF measurement. Coherence function is given by:

\[ \gamma_{pq}^2 = \frac{|GXF_{pq}|^2}{GFF_{qp}GXX_{pp}} \quad (46) \]

When the coherence function is equal to one, two signals are completely related. If the coherence function between two input signals is one, the two inputs are completely related. This would hinder the estimation of the FRF matrix.

**Multiple Inputs FRF estimation**

![Multiple Inputs FRF estimation](image)

**Figure (5) FRF representation for MDoF**
Calculate cross spectra matrix:

\[
\begin{bmatrix}
GXF
\end{bmatrix} = \{X\} \{F\}^H = \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_N
\end{bmatrix}
\begin{bmatrix}
F_1^* \\
F_2^* \\
\vdots \\
F_M^*
\end{bmatrix} = \begin{bmatrix}
GXF_{11} & GXF_{12} & \cdots & GXF_{1M} \\
GXF_{21} & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \\
GXF_{N1} & \cdots & \cdots & GXF_{NM}
\end{bmatrix}
\] (47)

Calculate input cross spectra matrix:

\[
\begin{bmatrix}
GFF
\end{bmatrix} = \{F\} \{F\}^H = \begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_M
\end{bmatrix}
\begin{bmatrix}
F_1^* \\
F_2^* \\
\vdots \\
F_M^*
\end{bmatrix} = \begin{bmatrix}
GFF_{11} & GFF_{12} & \cdots & GFF_{1M} \\
GFF_{21} & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \\
GFF_{M1} & \cdots & \cdots & GFF_{MM}
\end{bmatrix}
\] (48)

Where \( \{ \}^H \) denotes complex conjugate transpose matrix (Hermitian)

H₁ algorithm minimizes noise at outputs:

\[
[H] = [GXF][GFF]^{-1}
\] (49)

H₂ algorithm minimizes noise at inputs:

\[
[H] = [GXX][GFX]^{-1}
\] (50)

Fixed location of response with roving excitation is used with hammer excitation and provides a row of FRF matrix.

Fixed exciter with roving response sensor is used when a shaker is utilized as force exciter and the result is a column of FRF matrix.

**Types of Excitation**

1. Slow swept sine: periodic deterministic signal with frequency that is integer multiples of FFT frequency resolution.

2. Periodic chirp: deterministic signal where frequency is rapidly swept from minimum to maximum frequency in a single observation time period T.

3. Random excitation:
- Pure Random
- Pseudo Random
- Periodic Random
- Burst Random
- Slow Random
- Hybrid Random

Extracting Modal Parameters from Operating Data

Modal parameters refer to:

1. Modal frequencies
2. Modal damping matrix
3. Mode shape

One Column or One Row

Since all mode shapes are contained in every column and row, then one row or one column of FRF data is enough to find modal parameters. Suppose a 3 degree of freedom system with structural damping. And let the configuration used in SIMO with force applied at node 1.

\[
\begin{align*}
\varphi_1 &= \sum_{m=1}^{3} \left( \frac{\phi_1 m \phi_{1m}}{\lambda_m^2 - \omega^2} \right), \\
a_{11}(\omega) &= \sum_{m=1}^{3} \left( \frac{\phi_1 m \phi_{1m}}{\lambda_m^2 - \omega^2} \right), \\
a_{21}(\omega) &= \sum_{m=1}^{3} \left( \frac{\phi_2 m \phi_{1m}}{\lambda_m^2 - \omega^2} \right), \\
a_{31}(\omega) &= \sum_{m=1}^{3} \left( \frac{\phi_3 m \phi_{1m}}{\lambda_m^2 - \omega^2} \right) \\
\end{align*}
\]

The first column contains the mode shape multiplied by one of its components (the first element). The resonance frequencies and modal damping parameters can be found from any one of the column FRF or all of them.

Frequency Domain Analysis

For frequency domain modal analysis, the mathematical model is representative of the FRF data from measurement. However, in reality, the structure has infinite degree of freedom and modal frequencies. Assuming an \( M \) degree of freedom, for structural damping case:

\[
a_{ij}(\omega) = \sum_{m=1}^{M} \left( \frac{m \lambda_m A_{ij}}{\lambda_m^2 - \omega^2} \right) + \delta_{ij}(\omega) \tag{52}
\]

Here, \( \delta_{ij}(\omega) \) is the residual term which is usually made linear function so that modal analysis can be carried out without much inaccuracy due to invisible modes.
The purpose of modal analysis curve fitting is to reduce the error between the estimated FRF and the measured FRF. This depends not only on the accuracy of the measured data but also on the validity of the assumed model.

**Detection of Modes form FRF Matrix**

The identification of modal frequencies (vibration modes) prior to analysis needs to be based on several FRF curves. A convenient solution is to plot all the FRF curves in dB scales together. This will ensure that vibration modes become more prominent in the total FRF plot.

The phase plot of FRF can provide additional information. For undamped and lightly damped system, the phase change of 180° indicates presence of modal frequency. However, in damped system the phase change is not clear cut. Also, phase is very sensitive to measurement noise.

The Nyquist plot can provide valuable information in identifying modes. A genuine mode should show the symptom of a circle in the Nyquist plot. A false resonance peak will simply show a straight line on the Nyquist plane.

**Extracting Modal Data from FRF Data-SDoF methods**

These methods are suitable for uncoupled or lightly coupled systems. All SDoF modal analysis methods are based on the fact that at the vicinity of a resonance, the FRF is dominated by the contribution of that vibration mode and the contributions of other vibration modes are negligible. The simplicity of the mathematical model of an SDoF system can then be used in the curve fitting to derive the modal parameters mode by mode.

1. Peak-picking method

It is the simplest SDoF method for modal analysis. This method can give quick results but it is suitable for lightly damped systems (due to approximation in damping estimation algorithm) and it requires good frequency resolution. The procedure is:

   (a) Estimate the natural frequency from the peak. The natural frequency of \( m \)th mode is the frequency of the peak amplitude.

   (b) Estimate damping from half-power principle. The half power points before and after resonance are identified with frequencies \( \omega_1 \) and \( \omega_2 \), so:

\[
\eta_m = \frac{\omega_2^2 - \omega_1^2}{\omega_m^2} \frac{\omega_2 - \omega_1}{\omega_m} \tag{53}
\]
\[ \xi_m = \frac{\omega_2^2 - \omega_1^2}{4\omega_m^2} - \frac{\omega_2 - \omega_1}{2\omega_m} \]  \hspace{1cm} (54)

(c) Estimate the modal constants as follows. At resonance the response is mainly due to the corresponding mode, hence:

For structural damping:
\[ A_m = \alpha_{max} \eta_m \omega_m^2 \]  \hspace{1cm} (55)

For viscous damping:
\[ A_m = 2\alpha_{max} \zeta_m \omega_m^2 \]  \hspace{1cm} (56)

2. Circle-fit method

This method is used more than peak-pick method and provides more accurate results. It is based on the circularity of the Nyquist plot of an SDoF FRF as seen in eq. (13).

If the \( m \)th mode is to be analyzed, we can single it out from the summation so that:

\[ a_{ij}(\omega) = \frac{m A_{ij}}{\lambda_m^2 - \omega^2} + \sum_{k=1}^{N} \left( \frac{k A_{ij}}{\lambda_k^2 - \omega^2} \right) \]  \hspace{1cm} (57)

If the summation term in eq. (57) can be approximated as a constant, then the circularity of the Nyquist plot will not change other than the circle will shift away by a constant.

\[ a_{ij}(\omega) = \frac{m A_{ij}}{\lambda_m^2 - \omega^2} + B_{ij} \]  \hspace{1cm} (58)

The derivation of the modal data from circle fitting possesses the same procedure as the peak-picking method, i.e. from natural frequency to damping loss factor and to modal constant. However, the accuracy is improved.

It can be shown that the natural frequency \( \omega_m \) is the location where the maximum arc change occurs on the Nyquist circle.
From fig. (6) and remembering eq. (6b), it can be shown that:

\[
\tan \frac{\theta}{2} = \tan \delta = \frac{\text{Im}(\alpha)}{\text{Re}(\alpha)} = \frac{\omega_m^2 - \omega^2}{\eta_m \omega_m^2}
\]  

(59)

When choosing two angles on the circle; before and after resonance \(\omega_m\), the following equation can be found to calculate damping loss factor accurately:

\[
\eta_m = \frac{\omega_m^2 - \omega^2}{\omega_m^2} \left( \frac{1}{\tan \frac{\theta_1}{2} + \tan \frac{\theta_2}{2}} \right)
\]  

(60)

The modal constant is estimated from the circle diameter as follows:

\[
|m A_{ij}| = m D_{ij} \eta_m \omega_m^2
\]  

(61)

The phase angle of \(m A_{ij}\) is estimated from the location of the natural frequency.

The circle fit method can be summarized in the following steps:

1. select a vibration mode from the resonance peak of the receptance FRF
2. select some data points of the receptance FRF on the Nyquist plane
3. fit the best circle using selected points
4. locate the natural frequency of the mode from the maximum arc rate change and estimate the phase angle for the modal constant
5. estimate the damping loss factor

Figure (6) Nyquist plot of a single FRF
(6) determine the modulus of the modal constant from the diameter of the circle and its phase from the location of the natural frequency.

**Note:** According to eq. (57), the circle fit may be iteratively repeated by evaluating the constant $\sum_{k=1}^{N} \left( \frac{k A_{ij}}{\lambda_k^2 - \omega^2} \right)$ for the known modes. New set of data can be evaluated each iteration and compared with previous set. Iteration can be stopped when convergence is met.

### 3. Least Squares Curve Fitting using SDoF model

This method aims to find the best estimates of modal data of a certain mode that minimizes a defined error function. This function converts the curve fitting of a nonlinear mathematical model into a weighted linear model. Such a conversion makes the curve fitting simpler and the solution easier to derive.

Using the mobility FRF for a given SDoF:

$$Y_{ij}(\omega) = \frac{r A_{ij}}{\lambda_r^2 - \omega^2} \omega + \sum_{k=1 \atop k \neq r}^{N} \left( \frac{k A_{ij}}{\lambda_k^2 - \omega^2} \omega \right)$$  \hspace{1cm} (62)

The error between the measured $\tilde{Y}_{ij}(\omega)$ and estimated mobility FRFs for $r$th mode is given by:

$$E = \frac{r A_{ij}}{\lambda_r^2 - \omega^2} \omega + \sum_{k=1 \atop k \neq r}^{N} \left( \frac{k A_{ij}}{\lambda_k^2 - \omega^2} \omega \right) - \tilde{Y}_{ij}(\omega) = \frac{r A_{ij}}{\lambda_r^2 - \omega^2} \omega + R(\omega)$$  \hspace{1cm} (63)

Where $R(\omega)$ represent the other terms of estimated FRF minus the measured FRF. The curve fitting aims to minimize the error and find the best values for modal parameters.

### Extracting Modal Data from FRF Data-MDoF method

The idea of the rational fraction polynomial curve fitting method is to express an FRF in terms of rational fraction polynomials, and through numerical manipulations, the coefficients of these polynomials can be identified. These coefficients are post processed to obtain modal parameters.
Here, the degree of denominator \( n \) is greater than the degree of the numerator \( m \). The error function to be minimized is the difference between the measured and assumed model.

**Mode Shape Extraction from Modal Constants**

For a driving-point measurement, the FRF can be written as:

\[
a_{kk}(\omega) = \sum_{r=1}^{\text{mod} es} \left( \frac{r \cdot A_{kk}}{\lambda_r^2 - \omega^2} \right) = \sum_{r=1}^{\text{mod} es} \left( \frac{\phi_{kr}^2}{\lambda_r^2 - \omega^2} \right)
\]  

(65)

Hence:

\[
\phi_{kr} = \sqrt{r \cdot A_{kk}}
\]

and \( \phi_{jr} = \frac{r \cdot A_{jk}}{\phi_{kr}} \)  

(66)

Since all modes are contained on one column or one row, one driving point is enough to calculate all the required mode shape data. However, this represents the barest minimum requirement of modal data estimation. In many cases, it is important to measure more than a single column (or row) to provide additional data for checking or to replace poor data and make sure that no mode is missed due to improper selection of excitation location. For example, when a nodal point for \( r \)th mode is selected for excitation, then \( \phi_{kr} \) will be zero for that mode and consequently all modal constants will be zero for that mode.

**Post Processing and Optimization**

Once all the selected FRF curves have been measured and individually analyzed, there remains a further stage of processing to be done. Using any of the single-curve modal analysis methods, we shall find ourselves in possession of a set of tables of modal properties containing rather more data than we are seeking. In particular, we shall have determined many separate estimates for the natural frequency and damping factor of each mode of interest as these parameters are extracted afresh from each FRF curve in the measured set. In theory, all such estimates should be identical but in practice they seldom are, and we must find a way to reduce them to the single value for
each property which theory demands. The same thing is said about mode shape when we use more than one column (or one row).

The simplest procedure is simply to average all the individual estimates to obtain mean values for modal frequencies and damping factors. Sometimes we apply different weighting factors during averaging according to data reliability. For example, it is possible to attach a quality factor to each curve-fit parameter extraction.

**Quality of Modal Analysis**

**Modal Assurance Criterion (MAC)**

The modal assurance criterion was developed as a vector correlation tool to check the similarity or consistency between different modal vectors estimated from an experimental modal test. This tool is extended to cover comparisons with analytical models as well. Because different experimental modal vectors can be extracted from different modal parameter estimation techniques, or from different rows or columns of the frequency response matrix, or from entirely different modal tests. The MAC determines the correlation (similarity) between two different sets of results. MAC between two modal vectors from set obtained from two different sets ($i$ and $j$):

$$MAC_{ij} = \frac{\{\phi\}^H_i \{\phi\}^H_j}{\{\phi\}^H_i \{\phi\}^H_i \{\phi\}^H_j \{\phi\}^H_j}$$

(67)

Where $H$ denotes Hermitian (conjugate transpose) of the vector. If MAC is close to 1, then the correlation (similarity) is high.

**Practical Aspects**

1. Set Max. Frequency of analysis at least double the expected max. resonance frequency in the system.
2. Set record length more than the decay time of vibration in hammer test with power-of-two value for the number of samples.
3. Use trigger to select proper points from the acquired signal. For impact hammer test, the trigger could be the pulse of force transducer. Some points prior to the pulse are required to obtain satisfactory results. The record length should cover the pre pulse and post pulse data.
4. Use averaging for the calculated FRFs to obtain reliable results. Averaging is done in calculating cross and auto spectra before dividing them to obtain FRF. In the hammer impact excitation, you will need more than one impact to do averaging. Four averages (or more) are enough.

**Modal Data versus Operating Deflection Data**

<table>
<thead>
<tr>
<th>Mode shapes</th>
<th>Operating deflection shapes</th>
</tr>
</thead>
<tbody>
<tr>
<td>o Requires FRF estimation, thus requires measurement of inputs</td>
<td>o No need for input data measurement, however synchronous sampling or a reference sensor must be used</td>
</tr>
<tr>
<td>o Many individual mode shapes each mode is related to resonance modal frequency</td>
<td>o It is sum of mode shapes. Contribution of each mode depends on the excitation.</td>
</tr>
<tr>
<td>o can give system parameters</td>
<td>o Depend on the loading as well as material properties and boundary conditions</td>
</tr>
<tr>
<td>o Do not depend on the loading</td>
<td>o Can answer the question “what is the actual motion of a certain DOF versus another”</td>
</tr>
<tr>
<td>o Depend on material properties and boundary conditions</td>
<td></td>
</tr>
<tr>
<td>o Can answer the question “what is the relative motion between a certain DOF versus another”</td>
<td></td>
</tr>
</tbody>
</table>