1. Introduction

The main challenge in BEM is the evaluation of singular integrals arises when the distance \( r \) from the source point to the point under consideration is very small. In the case of planar (straight) elements, analytical integration is found to be very useful, but for the curved elements it cannot be implemented due to higher complexity [2]. Thus, the alternative method is to use numerical integration. In fact, numerical integration fails to evaluate singular integrals unless special treatments are applied such as element sub-division, variable transformation, subtraction of singularity and regularization. This is because the value of integral will be highly suspicious to the size of mesh used in integration. Table-1 shows the order of singularity for different problems.

<table>
<thead>
<tr>
<th></th>
<th>Elastostatic/Potential</th>
<th>Elastodynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2D</td>
<td>3D</td>
</tr>
<tr>
<td>Fund. Variable</td>
<td>( \ln \left( \frac{1}{r} \right) )</td>
<td>( \frac{1}{r} )</td>
</tr>
<tr>
<td>Fund. Derivative</td>
<td>( \frac{1}{r} )</td>
<td>( \frac{1}{r^2} )</td>
</tr>
</tbody>
</table>

Table-1 Orders of singularities for different problems

2. Evaluation of system matrices for 2D Potential and Elastostatic problems

The following discussion will be focused on potential problems although it can be extended to 2D Elastostatic problems. Consider a quadratic element as shown in Fig.1 with source point coincides with node (1).

The shape functions are:

\[
\begin{align*}
\phi_1(\zeta) &= 0.5\zeta(\zeta - 1) \\
\phi_2(\zeta) &= 1 - \zeta^2 \\
\phi_3(\zeta) &= 0.5\zeta(\zeta + 1)
\end{align*}
\]

The coefficients for fundamental potential are:

\[
G^{(n)} = \int_{-1}^{1} \ln\left(\frac{1}{r}\right) \phi_n(\zeta) |J| d\zeta \quad (1)
\]

and that for fundamental potential derivative are:

\[
H^{(n)} = -\int_{-1}^{1} \frac{1}{r} \frac{\partial}{\partial n} \phi_n(\zeta) |J| d\zeta \quad (2)
\]

Now, it is clear that a singularity problem will arise when trying to evaluate \( G^{(1)} \) and \( H^{(1)} \) because the value of the shape function is one at the singular node, while for the other coefficients the value of shape functions is zero and, hence, the
singularity will be cancelled and direct Gauss quadrature can be used. Fortunately, \( H^{(1)} \) can be estimated using the concept of constant potential over the boundary (rigid body motion for elasticity), while a special treatment must be adopted to evaluate \( G^{(1)} \). The distance \( r \) from the source point (node 1) to any point on the element can be expressed as:

\[
r = (\zeta + 1) r_d, \quad \text{where } r_d = \sqrt{[0.5\zeta(x_1 + x_3 - 2x_2) + x_2 - x_1] + [0.5\zeta(y_1 + y_3 - 2y_2) + y_2 - y_1]}
\]

so,

\[
G^{(1)} = \int_{-1}^{1} \ln \left( \frac{1}{(\zeta + 1) r_d} \right) \phi_n(\zeta) |J| d\zeta = \int_{-1}^{1} \ln \left( \frac{1}{\zeta + 1} \right) f(\zeta) d\zeta - \int_{-1}^{1} \ln r_d f(\zeta) d\zeta \quad \text{... (3)}
\]

where \( f(\zeta) = \phi_n(\zeta) |J| \)

introducing a new variable \( \eta = (\zeta + 1)/2 \) \( \rightarrow \zeta = 2\eta - 1 \) and \( d\zeta = 2 d\eta \), the first integral of the right side can be written as:

\[
\int_{-1}^{1} \ln \left( \frac{1}{\zeta + 1} \right) f(\zeta) d\zeta = 2\int_{0}^{1} \ln \left( \frac{1}{\eta} \right) f(2\eta - 1) d\eta - \int_{-1}^{1} \ln 2 f(\zeta) d\zeta
\]

The first integral of the right side can be evaluated using special Gauss quadrature, while the second one can be added to the integral of \( \ln r_d \) in eq. (3). Note that \( G^{(2)} \) and \( G^{(3)} \) can be evaluated using either direct or alternative method, but in the later case the accuracy is better. A similar procedure can be followed to evaluate the coefficients when the source point coincides with node (2) or (3).

3. Singular Integrals

Singular integrals are generally defined by eliminating a small portion of integration interval (space, area) that posses singularity, and obtaining the limit when this portion tends to vanish;

\[
\int_{D} f(v) dv = \lim_{\varepsilon \to 0} \int_{D - D_{\varepsilon}} f(v) dv \quad \text{...(4)}
\]

If the limit of the above equation exists regardless of how \( \varepsilon \) tends to zero, it is said improper integral and the singularity is weak. When the integral exists only for a certain form of limit to be taken, then it is said to be exist in the sense of Cauchy Principal Value (CPV).

Consider the following integral: \( \int_{-a}^{b} \ln|x| dx \), where both \( a \) and \( b \) are positive

The singularity is located at \( x=0 \) so the integral will be divided into two parts:

\[
\int_{-a}^{b} \ln|x| dx = \lim_{\varepsilon_1 \to 0} \int_{-a}^{-\varepsilon_1} \ln|x| dx + \lim_{\varepsilon_2 \to 0} \int_{\varepsilon_2}^{b} \ln|x| dx
\]

\[
= \lim_{\varepsilon_1 \to 0} \left[ x \ln|x| - x \right]_{-a}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \left[ x \ln|x| - x \right]_{\varepsilon_2}^{b}
\]

\[
= \lim_{\varepsilon_1 \to 0} \left[ x \ln|x| - x \right]_{-a}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \left[ x \ln|x| - x \right]_{\varepsilon_2}^{b}
\]
The above integral is called improper since it exists independently of how \( \varepsilon_1 \) and \( \varepsilon_2 \) tend to zero. Consider the following integral:

\[
\int_{-a}^{b} \frac{1}{x} \, dx = \lim_{\varepsilon \to 0} \left( \int_{-a}^{\varepsilon_1} \frac{1}{x} \, dx + \lim_{\varepsilon_2 \to 0} \int_{\varepsilon_2}^{b} \frac{1}{x} \, dx \right)
\]

The above limit doesn’t exist unless \( \varepsilon_1 = \varepsilon_2 = \varepsilon \), which means a condition for the existence and uniqueness of the integral, so it exists in the sense of \( \text{CPV} \).

\[
\int_{-a}^{b} \frac{1}{x} \, dx = \ln \frac{b}{a} + \lim_{\varepsilon \to 0} \frac{\varepsilon_1}{\varepsilon_2} = \ln \frac{b}{a}
\]

one must distinguish between the functions \( 1/x \) and \( 1/|x| \), the integral of the latter function doesn’t exist neither as improper nor as \( \text{CPV} \);

\[
\int_{-a}^{b} \frac{1}{|x|} \, dx = \lim_{\varepsilon \to 0} \ln |x|_{-a}^{\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \ln |x|_{\varepsilon_2}^{b}
\]

\[
= \ln |a - \ln \varepsilon_1| + \lim_{\varepsilon_2 \to 0} \ln |b - \ln \varepsilon_2| = \ln ab - \lim_{\varepsilon_1, \varepsilon_2 \to 0} \ln \varepsilon_1 + \ln \varepsilon_2 = \infty
\]

(*) In BEM, the \( \text{CPV} \) integrals sometimes appear as two parts associated with the two adjacent segments connected by the singular point. The integral along one segment is not defined due to end-point singularity, but the sum of integrals along both segments will cancel out any singularity;

\[
I_1 = \lim_{\varepsilon \to 0} \int_{-a}^{\varepsilon} \frac{1}{x} \, dx = -\ln a + \lim_{\varepsilon \to 0} \ln \varepsilon
\]

\[
I_2 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{b} \frac{1}{x} \, dx = \ln b - \lim_{\varepsilon \to 0} \ln \varepsilon
\]

The non-singular part of each of the above integrals is called finite part;
From the above, the relation between CPV and finite parts can easily be understood.

**Other conditions for the existence of CPV**

Consider the integral 
$$
\int_{-\epsilon}^{\epsilon} \frac{f(x)}{|x|} \, dx
$$

which can be divided into two parts:

$$
\int_{-\epsilon}^{\epsilon} \frac{f(x)}{|x|} \, dx = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{f(x)}{x} \, dx + \lim_{\epsilon \to 0} \int_{\epsilon}^{\epsilon} \frac{f(x)}{x} \, dx,
$$

using integration by parts.

$$
\int_{-\epsilon}^{\epsilon} \frac{f(x)}{|x|} \, dx = \lim_{\epsilon \to 0} \left[ -\ln|x|f(x) \right]_{-\epsilon}^{\epsilon} + \lim_{\epsilon \to 0} \left[ \int_{-\epsilon}^{\epsilon} \ln|x|f'(x) \, dx \right] + \lim_{\epsilon \to 0} \left[ \int_{-\epsilon}^{\epsilon} \ln|x|f'(x) \, dx \right] + \lim_{\epsilon \to 0} \left[ \int_{\epsilon}^{\epsilon} \ln|x|f'(x) \, dx \right]
$$

Since $I_1$ and $I_2$ are regular and can be evaluated using the special Gauss quadrature, then the whole integral would exist if the following limit exists:

$$
\lim_{\epsilon \to 0} \left[ \int_{-\epsilon}^{\epsilon} \ln|x|f(x) \right] = \ln b f(b) + \ln a f(-a) - \ln \lim_{\epsilon \to 0} \left[ \int_{-\epsilon}^{\epsilon} \ln|x|f(x) \right]
$$

which exist in the following case [3]: $|f(\epsilon) + f(-\epsilon)| < A\epsilon^\alpha$ where $0 < \alpha < 1$, $A > 0$.

This condition is called Hölder continuity condition which is an intermediate state between continuity and differentiability of a function around a specific point.

If we have 
$$
\int_{-\epsilon}^{\epsilon} \frac{f(x)}{|x|} \, dx
$$

then the continuity condition would be:

$$
|f(\epsilon) - f(-\epsilon)| < A\epsilon^\alpha
$$

where $0 < \alpha < 1$, $A > 0$.

It is worth noting when $|f(\epsilon)| < A\epsilon^\alpha$ then the integral is improper and there is no need for CPV. In BEM, since we can express the distance $r$ in terms of the singularity root, the second form will be commonly encountered.

In BEM, the singular point may be at the end of the element under consideration, in this case the integral along that element exist only in the sense of finite part, however if we consider the effect of the adjacent element, the non-finite parts will be cancelled out. This is because during assembly process, the condition of continuity is imposed in the discrete system, so the overall shape function will satisfy the condition of continuity as shown in the figure [1].
Let's consider now a stronger singularity arising from integrating for example \( f(x)/x^2 \):

\[
\int_{-a}^{b} \frac{f(x)}{x^2} \, dx = \lim_{\epsilon_1 \to 0} \int_{-a}^{-\epsilon_1} \frac{f(x)}{x^2} \, dx + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{b} \frac{f(x)}{x^2} \, dx \quad \text{using integration by parts}
\]

\[
\int_{-a}^{b} \frac{f(x)}{x^2} \, dx = \lim_{\epsilon \to 0} \left\{ \left[ -\frac{f(x)}{x} \right]_{-a}^{\epsilon} - \int_{-a}^{\epsilon} f'(x) \, dx \right\} + \lim_{\epsilon \to 0} \left\{ \left[ -\frac{f(x)}{x} \right]_{\epsilon}^{b} + \int_{\epsilon}^{b} f'(x) \, dx \right\}
\]

The two integrals in the right side exist in the condition of \( |f'(\epsilon) - f'(\epsilon)| < A\epsilon^a \) which means the first derivative satisfies Hölder continuity condition. So the challenge now is the following limit:

\[
\lim_{\epsilon \to 0} \left[ \frac{f(-\epsilon)}{\epsilon} - \frac{f(-a)}{a} + \frac{f(b)}{b} - \frac{f(\epsilon)}{\epsilon} \right] = \frac{f(b)}{b} - \frac{f(-a)}{a} + \lim_{\epsilon \to 0} \left[ \frac{f(-\epsilon) - f(\epsilon)}{\epsilon} \right]
\]

this limit exists under the following condition:

\[ |f(\epsilon) - f(-\epsilon)| < B\epsilon^\gamma \quad \text{where} \quad 1 < \gamma < 2 \ , B > 0 \]

This condition is satisfied either when \( f(x) \) has second continuous derivative or at least its first derivative satisfies Hölder condition [1]. The above integral exists in the sense of Hadamard Principal Value.

(*) In fact, for any degree of singularity \( m \), we can show that the condition of existence of the CPV integral is given by:

\[ |f(\epsilon) - f(-\epsilon)| < A\epsilon^\beta \quad \text{where} \quad (m-1) < \beta < m \ , A > 0 \]

(*) Integration by parts can be used to separate the non-finite parts of a singular integration to obtain a numerically integrable formula.

Example: Consider a quadratic straight element in which the singular point is node (1) as shown in figure. Suppose we'd like to calculate the coefficient of the field variable at node (1) and the singularity is of degree 2; let the length of the element be \( 2a \), so the distance \( r \) from the source point (node 1) to any point can be expressed as:

\[
r = a(\zeta + 1)
\]

\[
h^{(1)} = \int_{-1}^{1} \frac{1}{r^2} \phi_1(\zeta) \, d\Gamma = \int_{-1}^{1} \frac{1}{a^2(\zeta + 1)^2} \phi_1(\zeta) \, a \, d\zeta = \frac{1}{a} \int_{-1}^{1} \frac{1}{\zeta + 1} \phi_1(\zeta) \, d\zeta
\]

using analytical integration directly yields:

\[
h^{(1)} = 0.5 \left[ \frac{\zeta - \frac{2}{1+\zeta}}{3(1+\zeta)^2} \right]_{-1}^{1} = 0.5 \left[ (1-3 \ln 2) - (-1 - \frac{2}{0} - 3 \ln 0) \right]
\]

the finite part of the integral is \( 0.5 \frac{a}{a} (1 - 3 \ln 2) \)
Let's now use integration by parts to obtain a numerically integrable formula then using numerical integration and compare the results:

\[ h^{(1)} = \frac{1}{a} \int_{-1}^{1} \frac{1}{(\zeta + 1)^2} \phi(\zeta) d\zeta = \frac{1}{a} \int_{-1}^{1} \frac{1}{(\zeta + 1)^2} f(\zeta) d\zeta \]

where \( f(\zeta) = 0.5\zeta(\zeta - 1) = 0.5\zeta^2 - 0.5\zeta \Rightarrow f'(\zeta) = \zeta - 0.5 \Rightarrow f''(\zeta) = 1 \)

\[ h^{(1)} = \frac{1}{a} \left[ \left( \frac{-1}{\zeta + 1} f(\zeta) \right)_{-1}^{1} + \int_{-1}^{1} \frac{1}{\zeta + 1} f'(\zeta) d\zeta \right] \]

\[ = \frac{1}{a} \left[ \left( \frac{-f(1)}{2} + \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} + (\ln|\xi| + 1)f'(\xi) \right)_{-1}^{1} - \int_{-1}^{1} \ln|\xi| + 1 f''(\xi) d\xi \right] \]

\[ = \frac{1}{a} \left[ \left( \frac{-f(1)}{2} + \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} + f'(1)\ln 2 - f'(-1)\ln 0 + \int_{-1}^{1} \left[ \frac{2}{\xi + 1} f''(\xi) d\xi - \ln 2 f''(1) - f''(-1) \right] \right] \]

\[ = \frac{1}{a} \left[ \left( \frac{-f(1)}{2} + \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} + f'(1)\ln 2 + 2 \int_{0}^{1} \left[ \frac{1}{\eta} f''(2\eta - 1) d\eta - \ln 2 f'(-1) - f'(-1) \right] \right] \]

\[ = \frac{1}{a} \left[ \left( \frac{-f(1)}{2} + \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} + f'(1)\ln 2 + 2 \int_{0}^{1} \left[ \frac{1}{\eta} f''(2\eta - 1) d\eta \right] \right] \]

where \( \eta = \frac{\zeta + 1}{2} \)

The integral in the right side of the above result can easily be evaluated by numerical (special Gauss quadrature) to give 1, so

\[ h^{(1)} = \frac{1}{a} \left[ 0 + \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} - 1.5 \ln 2 + 2 \right] \]

unless a special treatment is considered in evaluating \( \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} \) the result of integration will be different from that obtained by analytical method. This is obviously due to end-point singularity where \( f(\zeta) \) does not satisfy continuity condition without considering the adjacent element effect. We will extract a finite part from the above limit which makes the value of integration similar to that of analytical method, the proposed treatment is either by differentiating both the denominator and nominator with respect to \( \zeta \) or using variable transformation \( \eta = \zeta + 1 \);

\[ \lim_{\xi \to -1} \frac{f(\xi)}{\xi + 1} = \lim_{\xi \to -1} \frac{f'(\xi)}{1} = \lim_{\xi \to -1} (\xi - 0.5) = -1.5 \]

imposing the above value into the result of integration, one obtains:
\[ h^{(1)} = \frac{1}{a} \left( 0.5 - 1.5 \ln 2 \right) = \frac{0.5}{a} (1 - 3 \ln 2) \]

Let's now discuss the estimation of \( h^{(2)} \) and \( h^{(3)} \) which represent the coefficients of field variable at nodes (2) and (3) respectively using analytical and integration by parts methods;
For \( h^{(2)} \), \( f(\zeta) = \phi_2(\zeta) = 1 - \zeta^2 \Rightarrow f'(\zeta) = -2\zeta \Rightarrow f''(\zeta) = -2 \)
the analytical integration directly yields \( h^{(2)} = \frac{1}{a} \left[ \zeta + 2 \ln 1 + \zeta \right]_{1} \)
and the FP is \( h^{(2)} = \frac{1}{a} \left[ 2 - 2 \ln 2 \right] \)
the integration by parts yields:
\[
\begin{align*}
    h^{(2)} &= \frac{1}{a} \left[ \frac{-f(1)}{2} + \lim_{\zeta \to 1} \frac{f(\zeta)}{\zeta + 1} + f'(1) \ln 2 + 2 \int_{0}^{1} \ln |\eta| f''(2\eta - 1) d\eta \right] \\
    h^{(2)} &= \frac{1}{a} \left[ 0 + \lim_{\zeta \to 1} \frac{1 - \zeta^2}{\zeta + 1} + 2 \ln 2 - 4 \right]
\end{align*}
\]
we need to evaluate \( \lim_{\zeta \to 1} \frac{1 - \zeta^2}{\zeta + 1} \) by l'Hopitals' rule which directly gives:
\[
\lim_{\zeta \to 1} \frac{1 - \zeta^2}{\zeta + 1} = \lim_{\zeta \to 1} \frac{-2\zeta}{1} = 2
\]
then \( h^{(2)} = \frac{1}{a} (2 + 2 \ln 2 - 4) = \frac{1}{a} (-2 + 2 \ln 2) \)
which is the same result of analytical method. For \( h^{(3)} \) we will obtain identical results if we follow the same procedure after substituting the corresponding function \( f(\zeta) = \phi_3(\zeta) \) and using l'Hopitals' rule in dealing with the limit \( \lim_{\zeta \to 1} \frac{f(\zeta)}{\zeta + 1} \). It is clear that for \( h^{(2)} \) and \( h^{(3)} \), \( f(-1)=0 \) which makes application of l'Hopitals' rule quiet legal and directly yields the value of the above limit.

**Subtraction of Singularity Method**
Another easy to apply method which can be used to separate the infinite parts and obtain regular integration, which can be evaluated numerically, is to subtract Taylor's expansion from the function \( f(x) \). This will cancel out singularity under certain conditions which will be discussed soon.
For the singularity of order \( 1/x \), only the first term of Taylor's expansion is sufficient to cancel out the singularity, consider
\[
\int_{a}^{b} \frac{f(x)}{x - x_0} \, dx \quad \text{where } x_0 \text{ lies between } a \text{ and } b
\]
\[
\int_{a}^{b} \frac{f(x)}{x - x_0} \, dx = \int_{a}^{b} \frac{f(x) - f(x_0)}{x - x_0} \, dx + f(x_0) \int_{a}^{b} \frac{1}{x - x_0} \, dx
\]
The first integral in the right-hand side exist if the function \( f(x) \) has continuous first derivative at \( x_0 \) or at least satisfies Hölder continuity condition:
\[
|f(x) - f(x_0)| < A|x - x_0|^{\alpha}
\]
where \( 0 < \alpha < 1 \) and \( A > 0 \)

while the second integral exists as CPV:
\[
\int_{a}^{b} \frac{dx}{x - x_0} = \ln|x - x_0| \quad \int_{a}^{b} \frac{b - x_0}{x - x_0} = \ln\left|\frac{b - x_0}{x - x_0 - a}\right|
\]

For hyper-singular integration, \( 1/x^2 \), the first and second terms of Taylor's expansion are necessary to cancel out the singularity, but \( f(x) \) must either have continuous second derivative at \( x_0 \) or at least its first derivative satisfies Hölder condition;
\[
\int_{a}^{b} \frac{f(x)}{(x - x_0)^2} \, dx = \int_{a}^{b} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} \, dx + f(x_0) \int_{a}^{b} \frac{1}{(x - x_0)^2} \, dx + f'(x_0) \int_{a}^{b} \frac{1}{x - x_0} \, dx
\]

In the right-hand side, the first integral is regular under the mentioned condition, the third integral has already been discussed and the second integral exists in the sense of Hadmard principal value:
\[
\int_{a}^{b} \frac{1}{(x - x_0)^2} \, dx = -\left(\frac{1}{a - x_0} - \frac{1}{b - x_0}\right)
\]

4. Symmetry

In BEM, there are two main types of symmetry, Boundary Condition Symmetry and Geometrical Symmetry. The first type has the effect of reducing system matrices since only part of the body under consideration will be discretized. However, this requires also geometrical symmetry to be present. On the other hand, the second type will save CPU time by mapping the coefficients of system matrices along the symmetric parts. Our discussion will be focused on the second type.

In fact, geometrical symmetry may exist in many different manners according to the discretization process and shape of the body (or parts of it) under consideration. The following discussion is focused on quadratic element although it can be generalized to less or higher order elements.

The first type of geometrical symmetry is that exist within an element. When the element is symmetric about the inner node, a reduction of CPU time by 33% can be obtained. Due to symmetry, the block of coefficients obtained when node (3) is the singular point is in the mirror of coefficients obtained when node (1) is the singular point.

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The first type of geometrical symmetry is that exist within an element. When the element is symmetric about the inner node, a reduction of CPU time by 33% can be obtained. Due to symmetry, the block of coefficients obtained when node (3) is the singular point is in the mirror of coefficients obtained when node (1) is the singular point.
When an element is encountered with the same shape and dimensions of an already manipulated element, then the singular coefficients of all nodes can be mapped directly resulting in considerable reduction in CPU time.

When the orientations of a source point with respect to two or more elements are identical, then the coefficients block corresponding to this source point can be mapped from one to another element. The followings are examples to this case:

(*) When the whole circle is to be discretized (due to asymmetric B.C.), only the coefficients of two nodes need to be calculated since the other coefficients for the remaining nodes can be mapped directly, given that all elements are identical.

(*) When a square is discretized into a number of identical elements, only 1/8 of the total number of nodes plus one can be considered. In the figure shown, only three nodes numbered 1, 2 and 3 need to be considered. When the shape is rectangular, then 1/4 of the nodes plus one must be considered.

(*) For each individual part (line, arc) constituting the final shape of the body, and when this part is discretized into a number of identical elements, only one half of the nodes need to be considered in calculating the coefficients relating these nodes with the elements belonging to the same part.

To make the program take into consideration geometrical symmetry, it must read an identification vector for each node in addition to the coordinates any boundary conditions. This vector has a length equal to the number of elements and each value in this vector has double components, this first one refers to the identical node number and the second component refers to the element number which has already been manipulated. When the program encounters some specific values in the identification vector, it directly maps the coefficients from the corresponding node and element. However, this add some complexity to the programming and huge additional data to be entered unless an automatic process is adopted, thus it
ca be limited to those problems requiring repeated calculations of system matrices such as elastodynamic problems where reduction of CPU time is vital.

(*) Note about squaring function: it is worth to be mentioned that using repeated multiplication instead of powers has considerable effect on reducing CPU time. For example, during the process of forming system matrices, using direct multiplication of a quantity by itself instead of squaring it normally encountered in the calculation of Jacobians and radii has reduce the CPU time by 60-70%. This is due to the fact that, power function is calculated by taking logarithmic value of the number and multiply it by the exponent then taking inverse logarithm and this consumes a lot of CPU resources. On the other hand, multiplication is near real time process because it is done in the math-coprocessor which is built in the main processor for the nowadays computers. To study the effect of using direct multiplication, the procedure of calculating sub-matrices ($h$ and $g$) relating a source point with a specific element has been called 320,000 times (corresponding for example of calculating system matrices for 125 quadratic elements for 10 time steps in elastodynamics), the time required in both cases is reported in the following table, the CPU saving is 71%.

<table>
<thead>
<tr>
<th>Calculation Method</th>
<th>Execution Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squaring function</td>
<td>7.4</td>
</tr>
<tr>
<td>Direct multiplication</td>
<td>2.1</td>
</tr>
</tbody>
</table>

References